Learning Trajectories

Children follow natural developmental progressions in learning, developing mathematical ideas in their own way. Curriculum research has revealed sequences of activities that are effective in guiding children through these levels of thinking. These developmental paths are the basis for Building Blocks *learning trajectories. Learning trajectories have three parts: a mathematical goal, a developmental path through which children develop to reach that goal, and a set of activities matched to each of those levels that help children develop the next level. Thus, each learning trajectory has levels of understanding, each more sophisticated than the last, with tasks that promote growth from one level to the next. The* Building Blocks *Learning Trajectories give simple labels, descriptions, and examples of each level. Complete learning trajectories describe the goals of learning, the thinking and learning processes of children at various levels, and the learning activities in which they might engage. This document provides only the developmental levels.*

Learning Trajectories for Primary Grades Mathematics

Developmental Levels

Frequently Asked Questions (FAQ)

1. ***Why use learning trajectories?*** Learning trajectories allow teachers to build the *mathematics of children— the thinking of children as it develops naturally.* So, we know that all the goals and activities are within the developmental capacities of children. We know that each level provides a natural *developmental building block* to the next level. Finally, we know that the activities provide the *mathematical building blocks* for school success, because the research on which they are based typically involves higher-income children.

2. ***When are children "at" a level?*** Children are at a certain level when most of their behaviors reflect the thinking—ideas and skills—of that level. Often, they show a few behaviors from the next (and previous) levels as they learn.

3. ***Can children work at more than one level at the same time?*** Yes, although most children work mainly at one level or in transition between two levels (naturally, if they are tired or distracted, they may operate at a much lower level). Levels are not "absolute stages." They are "benchmarks" of complex growth that represent distinct ways of thinking. So, another way to think of them is as a sequence of different patterns of thinking. Children are continually learning, within levels and moving between them.

4. ***Can children jump ahead?*** Yes, especially if there are separate "sub-topics." For example, we have combined many counting competencies into one "Counting" sequence with sub-topics, such as verbal counting skills. Some children learn to count to 100 at age 6 after learning to count objects to 10 or more, some may learn that verbal skill earlier. The sub-topic of verbal counting skills would still be followed.

5. ***How do these developmental levels support teaching and learning?*** The levels help teachers, as well as curriculum developers, assess, teach, and sequence activities. *Teachers who understand learning trajectories and the developmental levels that are at their foundation are more effective and efficient.* Through planned teaching and also encouraging informal, incidental mathematics, teachers help children learn *at an appropriate and deep level.*

6. ***Should I plan to help children develop just the levels that correspond to my children's ages?*** No! The ages in the table are typical ages children develop these ideas. *But these are rough guides only*—children differ widely. Furthermore, the ages below are lower bounds of what children achieve without instruction. So, these are *"starting levels" not goals.* We have found that children who are provided high-quality mathematics experiences are capable of developing to levels one or more years beyond their peers.

Each column in the table below, such as "Counting," represents a main developmental progression that underlies the learning trajectory for that topic.

For some topics, there are "subtrajectories"—strands within the topic. In most cases, the names make this clear. For example, in Comparing and Ordering, some levels are about the "Comparer" levels, and others about building a "Mental Number Line." Similarly, the related subtrajectories of "Composition" and "Decomposition" are easy to distinguish. Sometimes, for clarification, subtrajectories are indicated with a note in italics after the title. For example, in Shapes, Parts and Representing are subtrajectories within the Shapes trajectory.

Clements, D. H., Sarama, J., & DiBiase, A.-M. (Eds.). (2004). *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education.* Mahwah, NJ: Lawrence Erlbaum Associates.

Clements, D. H., & Sarama, J. (in press). "Early Childhood Mathematics Learning." In F. K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning.* New York: Information Age Publishing.

Learning Trajectories

Developmental Levels for Counting

The ability to count with confidence develops over the course of several years. Beginning in infancy, children show signs of understanding number. With instruction and number experience, most children can count fluently by age 8, with much progress in counting occurring in kindergarten and first grade. Most children follow a natural developmental progression in learning to count with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
1–2	Pre-Counter	1	A child at the earliest level of counting may name some numbers meaninglessly. The child may skip numbers and have no sequence.
1–2	Chanter	2	At this level a child may sing-song numbers, but without meaning.
2	Reciter	3	At this level the child verbally counts with separate words, but not necessarily in the correct order.
3	Reciter (10)	4	A child at this level can verbally count to 10 with some correspondence with objects. They may point to objects to count a few items but then lose track.
3		5	At this level a child can keep one-to-one correspondence between counting words and objects—at least for small groups of objects laid in a line. A corresponder may answer "how many" by recounting the objects starting over with one each time.
4	Counter (Small Numbers)	6	*At around 4 years children begin to count meaningfully. They accurately count objects to 5 and answer the "how many" question with the last number counted. When objects are visible, and especially with small numbers, begins to understand cardinality.* These children can count verbally to 10 and may write or draw to represent 1–5.
4	Producer—Counter To (Small Numbers)	7	The next level after counting small numbers is to count out objects up to 5 and produce a group of four objects. When asked to show four of something, for example, this child can give four objects.
4–5	Counter (10)	8	This child can count structured arrangements of objects to 10. He or she may be able to write or draw to represent 10 and can accurately count a line of nine blocks and says there are 9. A child at this level can also find the number just after or just before another number, but only by counting up from 1.
5–6	Counter and Producer—Counter to (10+)	9	Around 5 years of age children begin to count out objects accurately to 10 and then beyond to 30. They can keep track of objects that have and have not been counted, even in different arrangements. They can write or draw to represent 1 to 10 and then 20 and 30, and can give the next number to 20 or 30. These children can recognize errors in others' counting and are able to eliminate most errors in one's own counting.

Age Range	Level Name	Level	Description
5–6	Counter Backward from 10	10	Another milestone at about age 5 is being able to count backwards from 10.
6–7	Counter from N (N+1, N−1)	11	Around 6 years of age children begin to count on, counting verbally and with objects from numbers other than 1. Another noticeable accomplishment is that children can determine immediately the number just before or just after another number without having to start back at 1.
6–7	Skip-Counting by 10s to 100	12	A child at this level can count by tens to 100. They can count through decades knowing that 40 comes after 39, for example.
6–7	Counter to 100	13	A child at this level can count by ones through 100, including the decade transitions from 39 to 40, 49 to 50, and so on, starting at any number.
6–7	Counter On Using Patterns	14	At this level a child keeps track of counting acts by using numerical patterns such as tapping as he or she counts.
6–7	Skip Counter	15	The next level is when children can count by 5s and 2s with understanding.
6–7	Counter of Imagined Items	16	At this level a child can count mental images of hidden objects.
6–7	Counter On Keeping Track	17	A child at this level can keep track of counting acts numerically with the ability to count up one to four more from a given number.
6–7	Counter of Quantitative Units	18	At this level a child can count unusual units such as "wholes" when shown combinations of wholes and parts. For example when shown three whole plastic eggs and four halves, a child at this level will say there are five whole eggs.
6–7	Counter to 200	19	At this level a child counts accurately to 200 and beyond, recognizing the patterns of ones, tens, and hundreds.
7+	Number Conserver	20	A major milestone around age 7 is the ability to conserve number. A child who conserves number understands that a number is unchanged even if a group of objects is rearranged. For example, if there is a row of ten buttons, the child understands there are still ten without recounting, even if they are rearranged in a long row or a circle.

Developmental Levels for Comparing and Ordering Numbers

Comparing and ordering sets is a critical skill for children as they determine whether one set is larger than another to make sure sets are equal and "fair." Prekindergartners can learn to use matching to compare collections or to create equivalent collections. Finding out how many more or fewer in one collection is more demanding than simply comparing two collections. The ability to compare and order sets with fluency develops over the course of several years. With instruction and number experience, most children develop foundational understanding of number relationships and place value at ages 4 and 5. Most children follow a natural developmental progression in learning to compare and order numbers with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Object	1	At this early level a child puts objects into one-to-one correspondence, but with only intuitive understanding of resulting equivalence. For example, a child may know that each carton has a straw, but doesn't necessarily know there are the same numbers of straws and cartons.
2	Perceptual Comparer	2	At the next level a child can compare collections that are quite different in size (for example, one is at least twice the other) and know that one has more than the other. If the collections are similar, the child can compare very small collections.
2–3	First-Second Ordinal Counter	3	A child at this level can identify the first and often second objects in a sequence.
3	Nonverbal Comparer of Similar Items	4	At this level a child can identify that different organizations of the same number of small groups are equal and different from other sets. (1–4 items).
3	Nonverbal Comparer of Dissimilar Items	5	At the next level a child can match small, equal collections of dissimilar items, such as shells and dots, and show that they are the same number.
4	Matching Comparer	6	As children progress they begin to compare groups of 1–6 by matching. For example, a child gives one toy bone to every dog and says there are the same number of dogs and bones.
4	Knows-to-Count Comparer	7	A significant step occurs when the child begins to count collections to compare. At the early levels children are not always accurate when larger collection's objects are smaller in size than the objects in the smaller collection. For example, a child at this level may accurately count two equal collections, but when asked, says the collection of larger blocks has more.
4	Counting Comparer (Same Size)	8	At the next level children make accurate comparisons via counting, but only when objects are about the same size and groups are small (about 1–5).
5	Counting Comparer (5)	9	As children develop their ability to compare sets, they compare accurately by counting, even when larger collection's objects are smaller. A child at this level can figure out how many more or less.
5	Ordinal Counter	10	At the next level a child identifies and uses ordinal numbers from "first" to "tenth." For example, the child can identify who is "third in line."
5	Counting Comparer	11	At this level a child can compare by counting, even when the larger collection's objects are smaller. For example, a child can accurately count two collections and say they have the same number even if one has larger objects.
5	Mental Number Line to 10	12	At this level a child uses internal images and knowledge of number relationships to determine relative size and position. For example, the child can determine whether 4 or 9 is closer to 6.
5	Serial Orderer to 6+	13	Children demonstrate development in comparing when they begin to order lengths marked into units (1–6, then beyond). For example, given towers of cubes, this child can put them in order, 1 to 6. Later the child begins to order collections. For example, given cards with one to six dots on them, puts in order.
6	Counting Comparer (10)	14	The next level can be observed when the child compares sets by counting, even when larger collection's objects are smaller, up to 10. A child at this level can accurately count two collections of 9 each, and says they have the same number, even if one collection has larger blocks.
6	Mental Number Line to 10	15	As children move into the next level they begin to use mental rather than physical images and knowledge of number relationships to determine relative size and position. For example, a child at this level can answer which number is closer to 6, 4, or 9 without counting physical objects.
6	Serial Orderer to 6+	16	At this level a child can order lengths marked into units. For example, given towers of cubes the child can put them in order.
7	Place Value Comparer	17	Further development is made when a child begins to compare numbers with place value understandings. For example, a child at this level can explain that "63 is more than 59 because six tens is more than five tens even if there are more than three ones."

Age Range	Level Name	Level	Description
7	Mental Number Line to 100	18	Children demonstrate the next level in comparing and ordering when they can use mental images and knowledge of number relationships, including ones embedded in tens, to determine relative size and position. For example, a child at this level when asked, "Which is closer to 45, 30 or 50?"says "45 is right next to 50, but 30 isn't."

Age Range	Level Name	Level	Description
8+	Mental Number Line to 1000s	19	About age 8 children begin to use mental images of numbers up to 1,000 and knowledge of number relationships, including place value, to determine relative size and position. For example, when asked, "Which is closer to 3,500—2,000 or 7,000?"a child at this level says "70 is double 35, but 20 is only fifteen from 35, so twenty hundreds, 2,000, is closer."

Developmental Levels for Recognizing Number and Subitizing (Instantly Recognizing)

The ability to recognize number values develops over the course of several years and is a foundational part of number sense. Beginning at about age 2, children begin to name groups of objects. The ability to instantly know how many are in a group, called *subitizing*, begins at about age 3. By age 8, with instruction and number experience, most children can identify groups of items and use place values and multiplication skills to count them. Most children follow a natural developmental progression in learning to count with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Small Collection Namer	1	The first sign of a child's ability to subitize occurs when the child can name groups of one to two, sometimes three. For example, when shown a pair of shoes, this young child says, "Two shoes."
3	Nonverbal Subitizer	2	The next level occurs when shown a small collection (one to four) only briefly, the child can put out a matching group nonverbally, but cannot necessarily give the number name telling how many. For example, when four objects are shown for only two seconds, then hidden, child makes a set of four objects to "match."
3	Maker of Small Collections	3	At the next level a child can nonverbally make a small collection (no more than five, usually one to three) with the same number as another collection. For example, when shown a collection of three, makes another collection of three.
4	Perceptual Subitizer to 4	4	Progress is made when a child instantly recognizes collections up to four when briefly shown and verbally names the number of items. For example, when shown four objects briefly, says "four."
5	Perceptual Subitizer to 5	5	The next level is the ability to instantly recognize briefly shown collections up to five and verbally name the number of items. For example, when shown five objects briefly, says "five."

Age Range	Level Name	Level	Description
5	Conceptual Subitizer to 5+	6	At the next level the child can verbally label all arrangements to five shown only briefly. For example, a child at this level would say, "I saw 2 and 2 and so I saw 4."
5	Conceptual Subitizer to 10	7	The next step is when the child can verbally label most briefly shown arrangements to six, then up to ten, using groups. For example, a child at this level might say, "In my mind, I made two groups of 3 and one more, so 7."
6	Conceptual Subitizer to 20	8	Next, a child can verbally label structured arrangements up to twenty, shown only briefly, using groups. For example, the child may say, "I saw three 5s, so 5, 10, 15."
7	Conceptual Subitizer with Place Value and Skip Counting	9	At the next level a child is able to use skip counting and place value to verbally label structured arrangements shown only briefly. For example, the child may say, "I saw groups of tens and twos, so 10, 20, 30, 40, 42, 44, 46 . . . 46!"
8+	Conceptual Subitizer with Place Value and Multiplication	10	As children develop their ability to subitize, they use groups, multiplication, and place value to verbally label structured arrangements shown only briefly. At this level a child may say, "I saw groups of tens and threes, so I thought, five tens is 50 and four 3s is 12, so 62 in all."

Developmental Levels for Composing Number

(Knowing Combinations of Numbers)

Composing and decomposing are combining and separating operations that allow children to build concepts of "parts" and "wholes." Most prekindergartners can "see" that two items and one item make three items. Later, children learn to separate a group into parts in various ways and then to count to produce all of the number "partners" of a given number. Eventually children think of a number and know the different addition facts that make that number. Most children follow a natural developmental progression in learning to compose and decompose numbers with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
4	Pre-Part-Whole Recognizer	1	At the earliest levels of composing a child only nonverbally recognizes parts and wholes. For example, When shown four red blocks and two blue blocks, a young child may intuitively appreciate that "all the blocks" include the red and blue blocks, but when asked how many there are in all, may name a small number, such as 1.
5	Inexact Part-Whole Recognizer	2	A sign of development in composing is that the child knows that a whole is bigger than parts, but does not accurately quantify. For example, when shown four red blocks and two blue blocks and asked how many there are in all, names a "large number," such as 5 or 10.
5	Composer to 4, then 5	3	The next level is that a child begins to know number combinations. A child at this level quickly names parts of any whole, or the whole given the parts. For example, when shown four, then one is secretly hidden, and then is shown the three remaining, quickly says "1" is hidden.
6	Composer to 7	4	The next sign of development is when a child knows number combinations to totals of seven. A child at this level quickly names parts of any whole, or the whole given parts and can double numbers to 10. For example, when shown six, then four are secretly hidden, and shown the two remaining, quickly says "4" are hidden.
6	Composer to 10	5	The next level is when a child knows number combinations to totals of 10. A child at this level can quickly name parts of any whole, or the whole given parts and can double numbers to 20. For example, this child would be able to say "9 and 9 is 18."

Developmental Levels for Adding and Subtracting

Learning single-digit addition and subtraction is generally characterized as "learning math facts." It is assumed that children must memorize these facts, yet research has shown that addition and subtraction have their roots in counting, counting on, number sense, the ability to compose and decompose numbers, and place value. Research has shown that learning methods for adding and subtracting with understanding is much more effective than rote memorization of seemingly isolated facts. Most children follow an observable developmental progression in learning to add and subtract numbers with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
1	Pre +/−	1	At the earliest level a child shows no sign of being able to add or subtract.
3	Nonverbal +/−	2	The first inkling of development is when a child can add and subtract very small collections nonverbally. For example, when shown two objects, then one object going under a napkin, the child identifies or makes a set of three objects to "match."
4	Small Number +/−	3	The next level of development is when a child can find sums for joining problems up to 3 + 2 by counting all with objects. For example, when asked, "You have 2 balls and get 1 more. How many in all?" counts out 2, then counts out 1 more, then counts all 3: "1, 2, 3, 3!"

Age Range	Level Name	Level	Description
5	Find Result +/−	4	**Addition** Evidence of the next level in addition is when a child can find sums for joining (you had 3 apples and get 3 more, how many do you have in all?) and part-part-whole (there are 6 girls and 5 boys on the playground, how many children were there in all?) problems by direct modeling, counting all, with objects. For example, when asked, "You have 2 red balls and 3 blue balls. How many in all?" the child counts out 2 red, then counts out 3 blue, then counts all 5. **Subtraction** In subtraction, a child at this level can also solve take-away problems by separating with objects. For example, when asked, "You have 5 balls and give 2 to Tom. How many do you have left?" the child counts out 5 balls, then takes away 2, and then counts the remaining 3.
5	Find Change +/−	5	**Addition** At the next level a child can find the missing addend (5 + __ = 7) by adding on objects. For example, when asked, "You have 5 balls and then get some more. Now you have 7 in all. How many did you get?" the child counts out 5, then counts those 5 again starting at 1, then adds more, counting "6, 7," then counts the balls added to find the answer, 2. **Subtraction** Compares by matching in simple situations. For example, when asked, "Here are 6 dogs and 4 balls. If we give a ball to each dog, how many dogs won't get a ball?" a child at this level counts out 6 dogs, matches 4 balls to 4 of them, then counts the 2 dogs that have no ball.
5	Make It N +/−	6	A significant advancement in addition occurs when a child is able to count on. This child can add on objects to make one number into another, without counting from 1. For example, when asked, "This puppet has 4 balls but she should have 6. Make it 6," puts up 4 fingers on one hand, immediately counts up from 4 while putting up two fingers on the other hand, saying, "5, 6" and then counts or recognizes the two fingers.
6	Counting Strategies +/−	7	The next level occurs when a child can find sums for joining (you had 8 apples and get 3 more . . .) and part-part-whole (6 girls and 5 boys . . .) problems with finger patterns or by adding on objects or counting on. For example, when asked "How much is 4 and 3 more?" the child answers "4 . . . 5, 6, 7 [uses rhythmic or finger pattern]. 7!" Children at this level also can solve missing addend (3 + __ = 7) or compare problems by counting on. When asked, for example, "You have 6 balls. How many more would you need to have 8?" the child says, "6, 7 [puts up first finger], 8 [puts up second finger]. 2!"
6	Part-Whole +/−	8	Further development has occurred when the child has part-whole understanding. This child can solve all problem types using flexible strategies and some derived facts (for example, "5 + 5 is 10, so 5 + 6 is 11"), sometimes can do start unknown (__ + 6 = 11), but only by trial and error. This child when asked, "You had some balls. Then you get 6 more. Now you have 11 balls. How many did you start with?" lays out 6, then 3 more, counts and gets 9. Puts 1 more with the 3, says 10, then puts 1 more. Counts up from 6 to 11, then recounts the group added, and says, "5!"
6	Numbers-in-Numbers +/−	9	Evidence of the next level is when a child recognizes that a number is part of a whole and can solve problems when the start is unknown (__ + 4 = 9) with counting strategies. For example, when asked, "You have some balls, then you get 4 more balls, now you have 9. How many did you have to start with?" this child counts, putting up fingers, "5, 6, 7, 8, 9." Looks at fingers, and says, "5!"
7	Deriver +/−	10	At the next level a child can use flexible strategies and derived combinations (for example, "7 + 7 is 14, so 7 + 8 is 15") to solve all types of problems. For example, when asked, "What's 7 plus 8?" this child thinks: 7 + 8 □ 7 + [7 + 1] □ [7 + 7] + 1 = 14 + 1 = 15. A child at this level can also solve multidigit problems by incrementing or combining tens and ones. For example, when asked "What's 28 + 35?" this child thinks: 20 + 30 = 50; +8 = 58; 2 more is 60, 3 more is 63. Combining tens and ones: 20 + 30 = 50. 8 + 5 is like 8 plus 2 and 3 more, so, it's 13—50 and 13 is 63.
8+	Problem Solver +/−	11	As children develop their addition and subtraction abilities, they can solve all types of problems by using flexible strategies and many known combinations. For example, when asked, "If I have 13 and you have 9, how could we have the same number?" this child says, "9 and 1 is 10, then 3 more to make 13. 1 and 3 is 4. I need 4 more!"
8+	Multidigit +/−	12	Further development is evidenced when children can use composition of tens and all previous strategies to solve multidigit +/− problems. For example, when asked, "What's 37 − 18?" this child says, "I take 1 ten off the 3 tens; that's 2 tens. I take 7 off the 7. That's 2 tens and 0 . . . 20. I have one more to take off. That's 19." Another example would be when asked, "What's 28 + 35?" thinks, 30 + 35 would be 65. But it's 28, so it's 2 less . . . 63.

Developmental Levels for Multiplying and Dividing

Multiplication and division builds on addition and subtraction understandings and is dependent upon counting and place value concepts. As children begin to learn to multiply they make equal groups and count them all. They then learn skip counting and derive related products from products they know. Finding and using patterns aids in learning multiplication and division facts with understanding. Children typically follow an observable developmental progression in learning to multiply and divide numbers with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Nonquantitive Sharer "Dumper"	1	Multiplication and division concepts begin very early with the problem of sharing. Early evidence of these concepts can be observed when a child dumps out blocks and gives some (not an equal number) to each person.
3	Beginning Grouper and Distributive Sharer	2	Progression to the next level can be observed when a child is able to make small groups (fewer than 5). This child can share by "dealing out," but often only between two people, although he or she may not appreciate the numerical result. For example, to share four blocks, this child gives each person a block, checks each person has one, and repeats this.
4	Grouper and Distributive Sharer	3	The next level occurs when a child makes small equal groups (fewer than 6). This child can deal out equally between two or more recipients, but may not understand that equal quantities are produced. For example, the child shares 6 blocks by dealing out blocks to herself and a friend 1 at a time.
5	Concrete Modeler ×/÷	4	As children develop, they are able to solve small-number multiplying problems by grouping—making each group and counting all. At this level a child can solve division/sharing problems with informal strategies, using concrete objects—up to twenty objects and two to five people—although the child may not understand equivalence of groups. For example, the child distributes twenty objects by dealing out two blocks to each of five people, then one to each, until blocks are gone.
6	Parts and Wholes ×/÷	5	A new level is evidenced when the child understands the inverse relation between divisor and quotient. For example, this child understands "If you share with more people, each person gets fewer."
7	Skip Counter ×/÷	6	As children develop understanding in multiplication and division they begin to use skip counting for multiplication and for measurement division (finding out how many groups). For example, given twenty blocks, four to each person, and asked how many people, the child skip counts by 4, holding up one finger for each count of 4. A child at this level also uses trial and error for partitive division (finding out how many in each group). For example, given twenty blocks, five people, and asked how many should each get, this child gives three to each, then one more, then one more.
8+	Deriver ×/÷	7	At the next level children use strategies and derived combinations and solve multidigit problems by operating on tens and ones separately. For example, a child at this level may explain "7 × 6, five 7s is 35, so 7 more is 42."
8+	Array Quantifier	8	Further development can be observed when a child begins to work with arrays. For example, given 7 × 4 with most of 5 × 4 covered, a child at this level may say, "There's eight in these two rows, and five rows of four is 20, so 28 in all."
8+	Partitive Divisor	9	The next level can be observed when a child is able to figure out how many are in each group. For example, given twenty blocks, five people, and asked how many should each get, a child at this level says "four, because 5 groups of 4 is 20."
8+	Multidigit ×/÷	10	As children progress they begin to use multiple strategies for multiplication and division, from compensating to paper-and-pencil procedures. For example, a child becoming fluent in multiplication might explain that "19 times 5 is 95, because twenty 5s is 100, and one less 5 is 95."

Developmental Levels for Measuring

Measurement is one of the main real-world applications of mathematics. Counting is a type of measurement, determining how many items are in a collection. Measurement also involves assigning a number to attributes of length, area, and weight. Prekindergarten children know that mass, weight, and length exist, but they don't know how to reason about these or to accurately measure them. As children develop their understanding of measurement, they begin to use tools to measure and understand the need for standard units of measure. Children typically follow an observable developmental progression in learning to measure with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
3	Length Quantity Recognizer	1	At the earliest level children can identify length as an attribute. For example, they might say, "I'm tall, see?"
4	Length Direct Comparer	2	In the next level children can physically align two objects to determine which is longer or if they are the same length. For example, they can stand two sticks up next to each other on a table and say, "This one's bigger." 
5	Indirect Length Comparer	3	A sign of further development is when a child can compare the length of two objects by representing them with a third object. For example, a child might compare length of two objects with a piece of string. Additional evidence of this level is that when asked to measure, the child may assign a length by guessing or moving along a length while counting (without equal length units). The child may also move a finger along a line segment, saying 10, 20, 30, 31, 32.
5	Serial Orderer to 6+	4	At the next level a child can order lengths, marked in one to six units. For example, given towers of cubes, a child at this level puts in order, 1 to 6.
6	End-to-End Length Measurer	5	At the next level the child can lay units end-to-end, although he or she may not see the need for equal-length units. For example, a child might lay 9-inch cubes in a line beside a book to measure how long it is.

Age Range	Level Name	Level	Description
7	Length Unit Iterater	6	A significant change occurs when a child can use a ruler and see the need for identical units.
7	Length Unit Relater	7	At the next level a child can relate size and number of units. For example, the child may explain, "If you measure with centimeters instead of inches, you'll need more of them, because each one is smaller."
8	Length Measurer	8	As children develop measurement ability they begin to measure, knowing the need for identical units, the relationships between different units, partitions of unit, and zero point on rulers. At this level the child also begins to estimate. The child may explain, "I used a meter stick three times, then there was a little left over. So, I lined it up from 0 and found 14 centimeters. So, it's 3 meters, 14 centimeters in all."
8	Conceptual Ruler Measurer	9	Further development in measurement is evidenced when a child possesses an "internal" measurement tool. At this level the child mentally moves along an object, segmenting it, and counting the segments. This child also uses arithmetic to measure and estimates with accuracy. For example, a child at this level may explain, "I imagine one meterstick after another along the edge of the room. That's how I estimated the room's length is 9 meters."

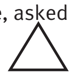
Developmental Levels for Recognizing Geometric Shapes

Geometric shapes can be used to represent and understand objects. Analyzing, comparing, and classifying shapes helps create new knowledge of shapes and their relationships. Shapes can be decomposed or composed into other shapes. Through their everyday activity, children build both intuitive and explicit knowledge of geometric figures. Most children can recognize and name basic two-dimensional shapes at 4 years of age. However, young children can learn richer concepts about shape if they have varied examples and nonexamples of shape, discussions about shapes and their characteristics, a wide variety of shape classes, and interesting tasks. Children typically follow an observable developmental progression in learning about shapes with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Shape Matcher—	1	The earliest sign of understanding shape is when a child can match basic shapes (circle, square, typical triangle) with the same size and orientation. Example: Matches to . A sign of development is when a child can match basic shapes with different sizes. Example: Matches to . The next sign of development is when a child can match basic shapes with different orientations. Example: Matches to .
3	Shape Prototype Recognizer and Identifier	2	A sign of development is when a child can recognize and name prototypical circle, square, and, less often, a typical triangle. For example, the child names this a square . Some children may name different sizes, shapes, and orientations of rectangles, but also accept some shapes that look rectangular but are not rectangles. Children name these shapes "rectangles" (including the non-rectangular parallelogram).
3	Shape Matcher—More Shapes	3	As children develop understanding of shape, they can match a wider variety of shapes with the same size and orientation. —4 Matches wider variety of shapes with different sizes and orientations. Matches these shapes . —5 Matches combinations of shapes to each other. Matches these shapes .
4	Shape Recognizer—Circles, Squares, and Triangles	4	The next sign of development is when a child can recognize some nonproto-typical squares and triangles and may recognize some rectangles, but usually not rhombi (diamonds). Often, the child doesn't differentiate sides/corners. The child at this level may name these as triangles .

Age Range	Level Name	Level	Description
4	Constructor of Shapes from Parts – Looks Like	5	A significant sign of development is when a child represents a shape by making a shape "look like" a goal shape. For example, when asked to make a triangle with sticks, the child creates the following .
5	Shape Recognizer—All Rectangles	6	As children develop understanding of shape, they recognize more rectangle sizes, shapes, and orientations of rectangles. For example, a child at this level correctly names these shapes "rectangles" .
5	Side Recognizer	7	A sign of development is when a child recognizes parts of shapes and identifies sides as distinct geometric objects. For example, when asked what this shape is , the child says it is a quadrilateral (or has four sides) after counting and running a finger along the length of each side.
5	Angle Recognizer	8	At the next level a child can recognize angles as separate geometric objects. For example, when asked, "Why is this a triangle," says, "It has three angles" and counts them, pointing clearly to each vertex (point at the corner).
5	Shape Recognizer	9	As children develop they are able to recognize most basic shapes and prototypical examples of other shapes, such as hexagon, rhombus (diamond), and trapezoid. For example, a child can correctly identify and name all the following shapes.
6	Shape Identifier	10	At the next level the child can name most common shapes, including rhombi, "ellipses-is-not-circle." A child at this level implicitly recognizes right angles, so distinguishes between a rectangle and a parallelogram without right angles. Correctly names all the following shapes:
6	Angle Matcher	11	A sign of development is when the child can match angles concretely. For example, given several triangles, finds two with the same angles by laying the angles on top of one another.

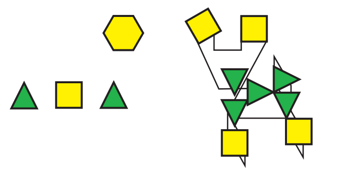
Learning Trajectories

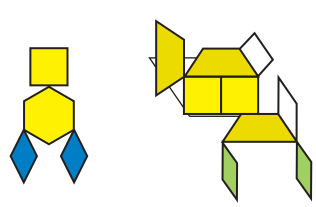
Age Range	Level Name	Level	Description
7	Parts of Shapes Identifier	12	At the next level the child can identify shapes in terms of their components. For example, the child may say, "No matter how skinny it looks, that's a triangle because it has three sides and three angles." 
7	Constructor of Shapes from Parts Exact	13	A significant step is when the child can represent a shape with completely correct construction, based on knowledge of components and relationships. For example, asked to make a triangle with sticks, creates the following: 
8	Shape Class Identifier	14	As children develop, they begin to use class membership (for example, to sort), not explicitly based on properties. For example, a child at this level may say, "I put the triangles over here, and the quadrilaterals, including squares, rectangles, rhombi, and trapezoids, over there."
8	Shape Property Identifier	15	At the next level a child can use properties explicitly. For example, a child may say, "I put the shapes with opposite sides parallel over here, and those with four sides but not both pairs of sides parallel over there."

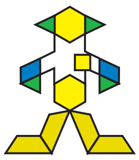
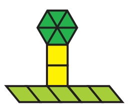
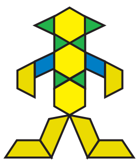
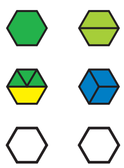
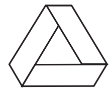
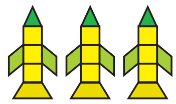
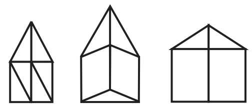
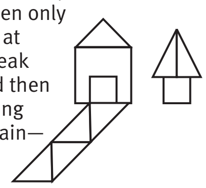
Age Range	Level Name	Level	Description
8	Angle Size Comparer	16	The next sign of development is when a child can separate and compare angle sizes. For example, the child may say, "I put all the shapes that have right angles here, and all the ones that have bigger or smaller angles over there."
8	Angle Measurer	17	A significant step in development is when a child can use a protractor to measure angles.
8	Property Class Identifier	18	The next sign of development is when a child can use class membership for shapes (for example, to sort or consider shapes "similar") explicitly based on properties, including angle measure. For example, the child may say, "I put the equilateral triangles over here, and the right triangles over here."
8	Angle Synthesizer	19	As children develop understanding of shape, they can combine various meanings of angle (turn, corner, slant). For example, a child at this level could explain, "This ramp is at a 45° angle to the ground."

Developmental Levels for Composing Geometric Shapes

Children move through levels in the composition and decomposition of two-dimensional figures. Very young children cannot compose shapes but then gain ability to combine shapes into pictures, synthesize combinations of shapes into new shapes, and eventually substitute and build different kinds of shapes. Children typically follow an observable developmental progression in learning to compose shapes with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

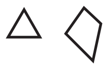
Age Range	Level Name	Level	Description
2	Pre-Composer	1	The earliest sign of development is when a child can manipulate shapes as individuals, but is unable to combine them to compose a larger shape. Make a Picture Outline Puzzle 
3	Pre-DeComposer	2	At the next level a child can decompose shapes, but only by trial and error. For example, given only a hexagon, the child can break it apart to make this simple picture by trial and error: 

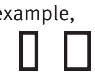
Age Range	Level Name	Level	Description
4	Piece Assembler	3	Around age 4 a child can begin to make pictures in which each shape represents a unique role (for example, one shape for each body part) and shapes touch. A child at this level can fill simple outline puzzles using trial and error. Make a Picture Outline Puzzle 

Age Range	Level Name	Level	Description
5	Picture Maker	4	As children develop they are able to put several shapes together to make one part of a picture (for example, two shapes for one arm). A child at this level uses trial and error and does not anticipate creation of the new geometric shape. The child can choose shapes using "general shape" or side length and fill "easy" outline puzzles that suggest the placement of each shape (but note below that the child is trying to put a square in the puzzle where its right angles will not fit). Make a Picture  Outline Puzzle 
5	Simple Decomposer	5	A significant step occurs when the child is able to decompose ("take apart" into smaller shapes) simple shapes that have obvious clues as to their decomposition.
5	Shape Composer	6	A sign of development is when a child composes shapes with anticipation ("I know what will fit!"). A child at this level chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. For example, in the outline puzzle below, all angles are correct, and patterning is evident. Make a Picture  Outline Puzzle 
6	Substitution Composer	7	A sign of development is when a child is able to make new shapes out of smaller shapes and uses trial and error to substitute groups of shapes for other shapes to create new shapes in different ways. For example, the child can substitute shapes to fill outline puzzles in different ways. 
6	Shape Decomposer (with Help)	8	As children develop they can decompose shapes by using imagery that is suggested and supported by the task or environment. For example, given hexagons, the child at this level can break it apart to make this shape: 
7	Shape Composite Repeater	9	The next level is demonstrated when the child can construct and duplicate units of units (shapes made from other shapes) intentionally, and understands each as being both multiple small shapes and one larger shape. For example, the child may continue a pattern of shapes that leads to tiling. 
7	Shape Decomposer with Imagery	10	A significant sign of development is when a child is able to decompose shapes flexibly by using independently generated imagery. For example, given hexagons, the child can break it apart to make shapes such as these: 
8	Shape Composer—Units of Units	11	Children demonstrate further understanding when they are able to build and apply units of units (shapes made from other shapes). For example, in constructing spatial patterns the child can extend patterning activity to create a tiling with a new unit shape—a unit of unit shapes that he or she recognizes and consciously constructs. For example, the child builds Ts out of four squares, uses four Ts to build squares, and uses squares to tile a rectangle.
8	Shape DeComposer with Units of Units	12	As children develop understanding of shape they can decompose shapes flexibly by using independently generated imagery and planned decompositions of shapes that themselves are decompositions. For example, given only squares, a child at this level can break them apart—and then break the resulting shapes apart again—to make shapes such as these: 

Developmental Levels for Comparing Geometric Shapes

As early as 4 years of age children can create and use strategies, such as moving shapes to compare their parts or to place one on top of the other for judging whether two figures are the same shape. From Pre-K to Grade 2 they can develop sophisticated and accurate mathematical procedures for comparing geometric shapes. Children typically follow an observable developmental progression in learning about how shapes are the same and different with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
3	"Same Thing" Comparer	1	The first sign of understanding is when the child can compare real-world objects. For example, the child says two pictures of houses are the same or different.
4	"Similar" Comparer	2	The next sign of development occurs when the child judges two shapes the same if they are more visually similar than different. For example, the child may say, "These are the same. They are pointy at the top." 
4	Part Comparer	3	At the next level a child can say that two shapes are the same after matching one side on each. For example, "These are the same" (matching the two sides). 
4	Some Attributes Comparer	4	As children develop they look for differences in attributes, but may examine only part of a shape. For example, a child at this level may say, "These are the same" (indicating the top halves of the shapes are similar by laying them on top of each other). 

Age Range	Level Name	Level	Description
5	Most Attributes Comparer	5	At the next level the child looks for differences in attributes, examining full shapes, but may ignore some spatial relationships. For example, a child may say, "These are the same." 
7	Congruence Determiner	6	A sign of development is when a child determines congruence by comparing all attributes and all spatial relationships. For example, a child at this level says that two shapes are the same shape and the same size after comparing every one of their sides and angles.
7	Congruence Superposer	7	As children develop understanding they can move and place objects on top of each other to determine congruence. For example, a child at this level says that two shapes are the same shape and the same size after laying them on top of each other.

Developmental Levels for Spatial Sense and Motions

Infants and toddlers spend a great deal of time exploring space and learning about the properties and relations of objects in space. Very young children know and use the shape of their environment in navigation activities. With guidance they can learn to "mathematize" this knowledge. They can learn about direction, perspective, distance, symbolization, location, and coordinates. Children typically follow an observable developmental progression in developing spatial sense with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
4	Simple Turner	1	An early sign of spatial sense is when a child mentally turns an object to perform easy tasks. For example, given a shape with the top marked with color, correctly identifies which of three shapes it would look like if it were turned "like this" (90 degree turn demonstrated) before physically moving the shape.

Age Range	Level Name	Level	Description
5	Beginning Slider, Flipper, Turner	2	The next sign of development is when a child can use the correct motions, but is not always accurate in direction and amount. For example, a child at this level may know a shape has to be flipped to match another shape, but flips it in the wrong direction.

Age Range	Level Name	Level	Description
6	Slider, Flipper, Turner	3	As children develop spatial sense they can perform slides and flips, often only horizontal and vertical, by using manipulatives. For example, a child at this level can perform turns of 45, 90, and 180 degrees and knows a shape must be turned 90 degrees to the right to fit into a puzzle.
7	Diagonal Mover	4	A sign of development is when a child can perform diagonal slides and flips. For example, a child at this level knows a shape must be turned or flipped over an oblique line (45 degree orientation) to fit into a puzzle.
8	Mental Mover	5	Further signs of development occur when a child can predict results of moving shapes using mental images. A child at this level may say, "If you turned this 120 degrees, it would be just like this one."

Developmental Levels for Patterning and Early Algebra

Algebra begins with a search for patterns. Identifying patterns helps bring order, cohesion, and predictability to seemingly unorganized situations and allows one to make generalizations beyond the information directly available. The recognition and analysis of patterns are important components of the young child's intellectual development because they provide a foundation for the development of algebraic thinking. Although prekindergarten children engage in pattern-related activities and recognize patterns in their everyday environment, research has revealed that an abstract understanding of patterns develops gradually during the early childhood years. Children typically follow an observable developmental progression in learning about patterns with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Pre-Patterner	1	A child at the earliest level does not recognize patterns. For example, a child may name a striped shirt with no repeating unit a "pattern."
3	Pattern Recognizer	2	At the next level the child can recognize a simple pattern. For example, a child at this level may say, "I'm wearing a pattern" about a shirt with black, white, black, white stripes.
3–4	Pattern Fixer	3	A sign of development is when the child fills in a missing element of a pattern. For example, given objects in a row with one missing, the child can identify and fill in the missing element.
4	Pattern Duplicator AB	3	A sign of development is when the child can duplicate an ABABAB pattern, although the child may have to work close to the model pattern. For example, given objects in a row, ABABAB, makes their own ABBABBABB row in a different location.
4	Pattern Extender AB	4	At the next level the child is able to extend AB repeating patterns.
4	Pattern Duplicator	4	At this level the child can duplicate simple patterns (not just alongside the model pattern). For example, given objects in a row, ABBABBABB, makes their own ABBABBABB row in a different location.
5	Pattern Extender	5	A sign of development is when the child can extend simple patterns. For example, given objects in a row, ABBABBABB, adds ABBABB to the end of the row.
7	Pattern Unit Recognizer	7	At this level a child can identify the smallest unit of a pattern. For example, given objects in a ABBAB_ BABB patterns, identifies the core unit of the pattern as ABB.

Developmental Levels for Classifying and Analyzing Data

Data analysis contains one big idea: classifying, organizing, representing, and using information to ask and answer questions. The developmental continuum for data analysis includes growth in classifying and counting to sort objects and quantify their groups. . . . Children eventually become capable of simultaneously classifying and counting, for example, counting the number of colors in a group of objects.

Children typically follow an observable developmental progression in learning about patterns with recognizable stages or levels. This developmental path can be described as part of a learning trajectory.

Age Range	Level Name	Level	Description
2	Similarity Recognizer	1	The first sign that a child can classify is when he or she recognizes, intuitively, two or more objects as "similar" in some way. For example, "that's another doggie."
2	Informal Sorter	2	A sign of development is when a child places objects that are alike on some attribute together, but switches criteria and may use functional relationships are the basis for sorting. A child at this level might stack blocks of the same shape or put a cup with its saucer.
3	Attribute Identifier	3	The next level is when the child names attributes of objects and places objects together with a given attribute, but cannot then move to sorting by a new rule. For example, the child may say, "These are both red."
4	Attribute Sorter	4	At the next level the child sorts objects according to a given attributes, forming categories, but may switch attributes during the sorting. A child at this stage can switch rules for sorting if guided. For example, the child might start putting red beads on a string, but switches to the spheres of different colors.
5	Consistent Sorter	5	A sign of development is when the child can sort consistently by a given attribute. For example, the child might put several identical blocks together.
6	Exhaustive Sorter	6	At the next level, the child can sort consistently and exhaustively by an attribute, given or created. This child can use terms "some" and "all" meaningfully. For example, a child at this stage would be able to find all the attribute blocks of a certain size and color.
6	Multiple Attribute Sorter	7	A sign of development is when the child can sort consistently and exhaustively by more than one attribute, sequentially. For example, a child at this level, can put all the attribute blocks together by color, then by shape.
7	Classifier and Counter	8	At the next level, the child is capable of simultaneously classifying and counting. For example, the child counts the number of colors in a group of objects.

Age Range	Level Name	Level	Description
7	List Grapher	9	In the early stage of graphing, the child graphs by simply listing all cases. For example, the child may list each child in the class and each child's response to a question.
8+	Multiple Attribute Classifier	10	A sign of development is when the child can intentionally sort according to multiple attributes, naming and relating the attributes. This child understands that objects could belong to more than one group. For example, the child can complete a two-dimensional classification matrix or forming subgroups within groups.
8+	Classifying Grapher	11	At the next level the child can graph by classifying data (e.g., responses) and represent it according to categories. For example, the child can take a survey, classify the responses, and graph the result.
8+	Classifier	12	At sign of development is when the child creates complete, conscious classifications logically connected to a specific property. For example, a child at this level gives definition of a class in terms of a more general class and one or more specific differences and begins to understand the inclusion relationship.
8+	Hierarchical Classifier	13	At the next level, the child can perform hierarchical classifications. For example, the child recognizes that all squares are rectangles, but not all rectangles are squares.
8+	Data Representer	14	Signs of development are when the child organizes and displays data through both simple numerical summaries such as counts, tables, and tallies, and graphical displays, including picture graphs, line plots, and bar graphs. At this level the child creates graphs and tables, compares parts of the data, makes statements about the data as a whole, and determines whether the graphs answer the questions posed initially.

Trajectory Progress Chart

Student's Name ______________________

Number

Age Range	Counting	Comparing and Ordering Number	Recognizing Number and Subitizing (instantly recognizing)	Composing Number (knowing combinations of numbers)	Adding and Subtracting	Multiplying and Dividing (sharing)
1 year	____ Pre-Counter ____ Chanter				____ Pre +/−	
2	____ Reciter	____ Object Corresponder ____ Perceptual Comparer	____ Small Collection Namer			____ Nonquantitative Sharer
3	____ Reciter (10) ____ Corresponder	____ First-Second Ordinal Counter ____ Nonverbal Comparer of Similar Items (1–4 items)	____ Nonverbal Subitizer ____ Maker of Small Collections		____ Nonverbal +/−	____ Beginning Grouper and Distributive Sharer
4	____ Counter (small numbers) ____ Producer (small numbers) ____ Counter (10)	____ Nonverbal Comparer of Dissimilar Items ____ Matching Comparer ____ Knows-to-Count Comparer ____ Counting Comparer (same size)	____ Perceptual Subitizer to 4	____ Pre-Part-Whole Recognizer	____ Small Number +/−	____ Grouper and Distributive Sharer
5	____ Counter and Producer (10+) ____ Counter Backward from 10	____ Counting Comparer (5) ____ Ordinal Counter	____ Perceptual Subitizer to 5 ____ Conceptual Subitizer to 5+ ____ Conceptual Subitizer to 10	____ Inexact Part-Whole Recognizer ____ Composer to 4, then 5	____ Find Result +/− ____ Find Change +/− ____ Make It *N* +/−	____ Concrete Modeler ×/÷
6	____ Counter from *N* (*N*+1, *N*−1) ____ Skip Counter by tens to 100 ____ Counter to 100 ____ Counter On Using Patterns ____ Skip Counter ____ Counter of Imagined Items ____ Counter On Keeping Track ____ Counter of Quantitative Units ____ Counter to 200	____ Counting Comparer (10) ____ Mental Number Line to 10 ____ Serial Orderer to 6+	____ Conceptual Subitizer to 20	____ Composer to 7 ____ Composer to 10	____ Counting Strategies +/− ____ Part-Whole +/−	____ Parts and Wholes ×/÷
7	____ Number Conserver ____ Counter Forward and Back	____ Place Value Comparer ____ Mental Number Line to 100	____ Conceptual Subitizer with Place Value and Skip Counting	____ Composer with Tens and Ones	____ Numbers-in-Numbers +/− ____ Deriver +/−	____ Skip Counter ×/÷
8+		____ Mental Number Line to 1,000s	____ Conceptual Subitizer with Place Value and Multiplication		____ Problem Solver +/− ____ Multidigit +/−	____ Deriver ×/÷ ____ Array Quantifier ____ Partitive Divisor ____ Multidigit ×/÷

Trajectory Progress Chart

Student's Name ______________________________

Geometry

Age Range	Shapes	Composing Shapes	Comparing Shapes	Motions and Spatial Sense	Measuring	Patterning	Classifying and Analyzing Data
2 years	____ Shape Matcher—Identical ____ —Sizes ____ —Orientations					____ Pre-Patterner	____ Similarity Recognizer ____ Informal Sorter
3	____ Shape Recognizer —Typical ____ Shape Matcher—More Shapes ____ —Sizes and Orientations ____ —Combinations	____ Pre-Composer ____ Pre-Decomposer	____ "Same Thing" Comparer		____ Length Quantity Recognizer	____ Pattern Recognizer	____ Attribute Identifier
4	____ Shape Recognizer—Circles, Squares, and Triangles+ ____ Constructor of Shapes from Parts—Looks Like Representing	____ Piece Assembler	____ "Similar" Comparer ____ Part Comparer ____ Some Attributes Comparer	____ Simple Turner	____ Length Direct Comparer	____ Pattern Fixer ____ Pattern Duplicator AB ____ Pattern Extender AB ____ Pattern Duplicator	____ Attribute Sorter
5	____ Shape Recognizer—All Rectangles ____ Side Recognizer ____ Angle Recognizer ____ Shape Recognizer—More Shapes	____ Picture Maker ____ Simple Decomposer ____ Shape Composer	____ Most Attributes Comparer	____ Beginning Slider, Flipper, Turner	____ Indirect Length Comparer	____ Pattern Extender	____ Consistent Sorter
6	____ Shape Identifier ____ Angle Matcher Parts	____ Substitution Composer ____ Shape Decomposer (with help)		____ Slider, Flipper, Turner	____ Serial Orderer to 6+ ____ End-to-End Length Measurer		____ Exhaustive Sorter ____ Multiple Attribute Sorter
7	____ Parts of Shapes Identifier ____ Constructor of Shapes from Parts—Exact Representing	____ Shape Composite Repeater ____ Shape Decomposer with Imagery	____ Congruence Determiner ____ Congruence Superposer	____ Diagonal Mover	____ Length Unit Iterater ____ Length Unit Relater	____ Pattern Unit Recognizer	____ Classifier and Counter ____ List Grapher
8+	____ Shape Class Identifier ____ Shape Property Identifier ____ Angle Size Comparer ____ Angle Measurer ____ Property Class Identifier ____ Angle Synthesizer	____ Shape Composer—Units of Units ____ Shape Decomposer with Units of Units	____ Congruence Representer	____ Mental Mover	____ Length Measurer ____ Conceptual Ruler Measurer		____ Multiple Attribute Classifier ____ Classifying Grapher ____ Classifier ____ Hierarchical Classifier ____ Data Representer